Topology and its Applications 46 (1992) 81-87
North-Holland

Free degrees of homeomorphisms and periodic maps on closed surfaces

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Received 17 December 1990
Revised 29 May 1991

Abstract

Wang, S., Free degrees of homeomorphisms and periodic maps on closed surfaces, Topology and its Applications 46 (1992) 81-87.

For each closed hyperbolic (orientable or nonorientable) surface F, we provide a positive integer $h(F)$ with properties that for any homeomorphism f on F, at least one of the iterates $f, f^2, \dots, f^{h(F)}$ has a fixed point and there is a periodic (orientation reversing) homeomorphism s on F such that $s^1, s^2, \dots, s^{h(F)-1}$ are all fixed point free. Those integers $h(F)$ are: $h(F_2)=4$, $h(F_g)=2g-2$, $g>2$, $h(N_3)=2$, $h(N_q)=q-2$, $q>3$. There is an interesting difference between our result and the Nielsen's result in the orientable category.

Keywords: Fixed point, closed surface, homeomorphism, periodic map.

AMS (MOS) Subj. Class.: Primary 55M20, 57A10.

Let M be a closed manifold and s be a homeomorphism on M. The free degree of s is the maximum positive integer m such that $s^1, s^2, \dots, s^{m-1}$ are all fixed point free. We call the maximum free degree of homeomorphisms on M, denoted by $h(M)$, the free degree of homeomorphisms on M.

By the result in [3], $h(M)$ exists if the Euler number of M is nonzero. If M is a differential manifold, by the well-known fact in differential topology, $h(M)$ exists if and only if the Euler number of M is nonzero.

We can define $p(M)$, the free degree of periodic homeomorphisms (but arbitrary periods) on M; $h_0(M)$, the free degree of orientation preserving homeomorphisms on M (if M is orientable); $p_0(M)$, the free degree of orientation preserving periodic homeomorphisms (but arbitrary periods) on M (if M is orientable) in the same way.

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* The author is partially supported by NSFC grants.

By the definitions, we have $p_0(M) \leq h_0(M) \leq h(M)$, $p_0(M) \leq p(M) \leq h(M)$. Let F_g be an orientable closed surface of genus g, $g \geq 2$, N_q be a nonorientable closed surface of genus q, $q \geq 3$. The result in this paper is

Theorem 1.

$$h(F_g) = p(F_g) = \begin{cases} 4, & \text{if } g = 2, \\ 2g-2, & \text{if } g > 2, \end{cases}$$

$$h(N_q) = p(N_q) = \begin{cases} 2, & \text{if } q = 3, \\ q-2, & \text{if } q > 3. \end{cases}$$

Remark. (1) Nielsen studied this problem in the forties in the orientable category. His result is

$$h_0(F_g) = \begin{cases} 2 \text{ or } 3, & \text{if } g = 2, \\ 2g-2, & \text{if } g > 2, \end{cases}$$

and

$$p_0(F_g) = \begin{cases} 2, & \text{if } g = 2, \\ g-1, & \text{if } g > 2. \end{cases}$$

(See [5], reviewed by Fox, that here $h_0(F_2)$ is 2 or 3 is still unknown.)

Thus it is a little bit surprising that the free degrees of homeomorphisms are realized by periodic maps in our theorem.

(2) The closed surfaces not included in Theorem 1 are the 2-sphere, torus, projective plane and Klein bottle. All these cases are well studied.

Now we give some definitions and results which will be used in the proof of Theorem 1.

(a) Let S be a closed hyperbolic orbifold with n singular points of indices $v_1, v_2, \ldots, v_n$. We denote the underlying space of S by $|S|$. The general Euler number of S, $\tilde{\chi}(S)$, is defined by the formula

$$\tilde{\chi}(S) = \chi(|S|) - \sum_{i=1}^{n} \left(1 - \frac{1}{v_i}\right);$$

here χ is the ordinary Euler number. The general fundamental group (or the fundamental group of orbifolds) of S is

$$\pi_1^0(S) = \begin{cases} \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid c_1^{v_1} = \cdots = c_n^{v_n} \\ \qquad = [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_n = 1 \rangle, \text{ or} \\ \langle a_1, \ldots, a_q, c_1, \ldots, c_n \mid c_1^{v_1} = \cdots = c_n^{v_n} = a_1^2 \cdots a_q^2 c_1 \cdots c_n = 1 \rangle. \end{cases}$$

If $n = 0$, then $\tilde{\chi}(S) = \chi(S) = \chi(|S|)$ and $\pi_1(|S|) = \pi_1^0(S)$. (See [6, Section 2].)

Any surjective homomorphism $\phi : \pi_1^0(S) \to Z_d$ determines a cyclic branched cover $p : \tilde{S} \to S$, here Z_d is a cyclic group of order d and $\tilde{S}$ is a hyperbolic orbifold with $\pi_1^0(\tilde{S}) = \text{kernel } \phi$ (see [1] or [4]). $\tilde{S}$ is a hyperbolic surface if and only if kernel ϕ is a surface group. kernel ϕ is a surface group if and only if c_i and $\phi(c_i)$ have the same order [4, Theorem 3].

(b) For any periodic map f on a closed surface F with $\chi(F)<0$, there is a hyperbolic metric on F such that f is an isometry under this metric.

Let f be an isometry of order d on a closed hyperbolic surface F with isolated periodic points only, then the quotient space $F/\langle f\rangle$ is a hyperbolic orbifold S and each singular point of index v_i is determined by an isolated periodic point orbit of f with length d/v_i.

Furthermore we have the Riemann-Hurwitz formula (see [4])

$$\frac{\tilde{\chi}(F)}{d}=\tilde{\chi}(S)=\chi(|S|)-\sum_{i=1}^{n}\left(1-\frac{1}{v_i}\right).$$

(c) Let $f: F\to F$ be a self map on a closed surface F. The Lefschetz number $L(f)$ is defined as $\sum_{i=0}^{2}(-1)^i\operatorname{trace}(f_{*i})$, here f_{*i} is a homomorphism on $H_i(F, Q)$ induced by f. The Lefschetz Fixed Point Theorem claims that if f is fixed point free, then $L(f)$ is zero (see [2]).

Proof of Theorem 1. The proof of Theorem 1 is elementary. It will be finished by two steps. The first step is to construct a periodic map with the required free degree on each closed surface. The constructions are inspired by our work [9]. The second step is to show the numbers in Theorem 1 are the upper bounds of the free degrees of homeomorphisms on closed surfaces. This step is to apply Fuller's idea (originated from Nielsen, see [2, 3]) to our concrete cases and to make a refinement.

Step 1. There are five cases.

Case 1: *Construction of a required map for* F_2.

Let the hyperbolic orbifold S be the RP^2 with two singular points of indices 2 and 3. A surjective homomorphism $\phi:\pi_1^0(S)=\langle a, c_1, c_2\,|\,a^2c_1c_2=c_1^3=c_2^2=1\rangle\to Z_{12}$ is defined by $\phi(c_1)=4$, $\phi(c_2)=6$, $\phi(a)=1$ in Z_{12}; here c_1 and c_2 are presented by orientable loops, a is presented by nonorientable loop. Obviously ϕ is surjective. So ϕ determines a cyclic branched covering $\tilde{S}\to S$ of degree 12. Since $\phi(c_i)$ and c_i have the same order, kernel ϕ is a surface group by (b). Pick any $\alpha\in$ kernel ϕ. Suppose the sums of powers of a, c_1, c_2 in α are k, l, m respectively. Now we must have $k+4l+6m=12j$ for some integer j, so k is even. Thus α is an orientable loop. It follows that $\tilde{S}$ is an orientable surface. From $\chi(\tilde{S})=-12\times(1-\frac{1}{2}-\frac{1}{3})=-2$, we know $\tilde{S}=F_2$ and the generator s of the deck group of covering is a map of order 12.

Since S has only two singular points of indices 2 and 3, we know that the periodic map s on $\tilde{S}$ has only two periodic orbits of length $\frac{12}{2}=6$ and $\frac{12}{3}=4$. So s, s^2, s^3 are all fixed point free.

Case 2: *Construction of a required map on* F_g, $g>2$, *g is even.*

Let the hyperbolic surface S be the connected sum of three RP^2. A surjective homomorphism $\phi:\pi_1^0(S)=\pi_1(S)=\langle a, b, c\,|\,aba^{-1}b^{-1}c^2=1\rangle\to Z_{2g-2}$ is defined by $\phi(a)=2$, $\phi(b)=2$, $\phi(c)=g-1$ in Z_{2g-2}, here a and b are presented by orientable loops and c is presented by a nonorientable loop. Since $g-1$ is odd, the largest common divisor $(g-1,2)=1$ and ϕ is surjective. Obviously, kernel ϕ is a surface group and ϕ determines a covering $\tilde{S}\to S$ of degree $2g-2$ without branched point.

Pick any $\alpha \in \text{kernel } \phi$. Suppose the sum of powers of a, b, c in α are k, l, m respectively. Now we must have $2k+2l+m(g-1)=j(2g-2)$ for some integer j. Since $g-1$ is odd, m is even, α is an orientable loop. It follows that $\tilde{S}$ is an orientable surface. From $\tilde{\chi}(\tilde{S})=\chi(\tilde{S})=(2g-2)\times\chi(S)=2-2g$, we know $\tilde{S}=F_g$ and the generator s_1 of the deck group of the covering is a map of order $2g-2$ with free degree $2g-2$.

Case 3: *Construction of a required map for F_g, $g>2$, g is odd.*

Let the hyperbolic orbifold S be the Klein bottle with one singular point of index 2. A surjective homomorphism $\phi:\pi_1^0(S)=\langle a_1, a_2, c \mid a_1^2a_2^2c=c^2=1\rangle\to Z_{4g-4}$ is defined by $\phi(c)=2g-2$, $\phi(a_1)=-1$, $\phi(a_2)=g$ in Z_{4g-4}, here α_1, α_2 are presented by nonorientable loops, c is presented by an orientable loop. Obviously ϕ is surjective. So ϕ determines a cyclic branched covering $\tilde{S}\to S$ of degree $4g-4$. Since $\phi(c)$ and c have the same order, kernel ϕ is a surface group by (b). Pick any $\alpha\in\text{kernel } \phi$. Suppose the sums of powers of a_1, a_2, c in α are k, l, m respectively. Now we must have $-k+lg+m(2g-2)=j(4g-4)$ for some integer j, i.e., $-k+l=j(4g-4)-m(2g-2)-l(g-1)$. Since g is odd, $k+l$ is even. Thus α is an orientable loop. It follows that $\tilde{S}$ is an orientable surface. From $\chi(\tilde{S})=(4g-4)\times\tilde{\chi}(S)=(4g-4)\times(-\frac{1}{2})=2-2g$, we know $\tilde{S}=F_g$ and the generator s of the deck group of branched covering is a map of order $4g-4$.

Since S has only one singular point of index 2, we know that the periodic map s on $\tilde{S}$ has only one periodic orbit of length $(4g-4)/2=2g-2$. So $s, s^2, \ldots, s^{2g-3}$ are all fixed point free.

Case 4: *Construction of an example on N_3.*

Let a, b be a basis for the first homology group of torus T. There is an orientation preserving periodic map ι' of order 6 determined by the map $(a,b)\mapsto(-b, a+b)$. Now $L(\iota')=1$, by the elementary fact in fixed point theory, ι' has only one fixed point x. Let D be an invariant small disk of ι' centered at x. The restriction of ι' on $S_1=T-\text{int } D$ is a map ι of the same order which is fixed point free. Now parameterize ∂S_1 by an angle $0\leq\theta\leq 2\pi$ such that the restriction of ι on ∂S_1 is a rotation of the angle $2\pi/6$.

Now we identify the antipodal points on ∂S_1, we get N_3 and ι induces a fixed point free map of order 6 on N_3.

Case 5: *Construction of an example on N_q, $q>3$.*

Pick an oriented essential simple closed curve c on torus T. Let s_1 be a rotation of torus T with angle $2\pi/(q-2)$ along this curve c and D be a small disk such that $D, s_1(D), \ldots, s_1^{q-3}(D)$ are disjoint. Let $S_1=T-\bigcup_{i=1}^{q-2} s_1^i(D)$. Identifying the antipodal points of each component of ∂S_1, we get N_q and s_1 induces a periodic map s of order $p-2$ such that $s, s^2, \ldots, s^{p-3}$ are all fixed point free.

Remark. (3) The example in Case 4 cannot be produced by using cyclic branched covering, since s^3 has a one-dimensional fixed point set.

Step 2. To show the numbers in Theorem 1 are the upper bounds of the free degrees on closed surfaces.

The fact that $h(N_q)'s$ and $h_0(F_g)'s$ are bounded by the numbers in Theorem 1 has been proved in [8] and [5] respectively. However we would like to prove the general case rather than to prove the case of orientation reversing homeomorphisms on orientable surfaces only for two reasons: (1) [8] and [5] are not commonly available or not easy to read; (2) the proof for the general case is only few lines longer than the proof of the special case.

Let f be a homeomorphism on a closed surface F with $\chi(F)<0$ such that $f, f^2, \ldots, f^n$ are all fixed point free.

Then we have

$$L(f^{-2})=L(f^{-1})=L(f)=L(f^2)=\cdots=L(f^n)=0. \tag{1}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the eigenvalues of an induced isomorphism f_{*1}, here s is the rank of $H_1(F, Q)$, then the characteristic polynomial of f_{*1} is

$$P(x)=(x-\lambda_1)\cdots(x-\lambda_s)=x^s-p_1x^{s-1}+\cdots+(-1)^s p_s, \tag{2}$$

here

$$p_s=\lambda_1\lambda_2\cdots\lambda_s=\pm 1. \tag{3}$$

For any nonzero integer, set $\iota_k=\sum_{i=1}^s \lambda_i^k$. Then

$$p_{s-1}=\sum_{i=1}^{s}\lambda_1\cdots\hat{\lambda}_i\cdots\lambda_s=\sum_{i=1}^{s}\frac{1}{\lambda_i}=\iota_{-1}, \tag{4}$$

here $\hat{\lambda}_i$ is the omission of λ_i, and

$$p_{s-2}=\sum_{1\le i<j\le s}\lambda_1\cdots\hat{\lambda}_i\cdots\hat{\lambda}_j\cdots\lambda_s=\sum_{1\le i<j\le s}\frac{1}{\lambda_i\lambda_j},$$

we have

$$\iota_{-2}=p_{s-1}^2-2p_{s-2}. \tag{5}$$

By Newton's formulas about elementary symmetric polynomials (see [7, Exercise 2, Section 29], for example), we have

$$\iota_1-p_1=0, \tag{6.1}$$

$$\iota_2-p_1\iota_1+2p_2=0, \tag{6.2}$$

$$\iota_3-p_1\iota_2+p_2\iota_1-3p_3=0, \tag{6.3}$$

and in general

$$\iota_k-p_1\iota_{k-1}+\cdots+(-1)^{k-1}p_{k-1}\iota_1+(-1)^k k p_k=0, \quad 1\le k\le s. \tag{6.k}$$

With all those elementary facts in hand, we are going to prove the second step.

Case 1: *Suppose f is a homeomorphism on F_g such that $f, \ldots, f^n$ are all fixed point free.* We show $n<2g$ and $n<2g-2$ if further $g>2$. Now $s=2g$.

If f is orientation reversing, then f^{2k-1} is a self map of degree -1 and f^{2k} is a self map of degree 1 for any integer k. From (1), linear algebra and the definition of the Lefschetz number, we have

$$\iota_{2k}=\sum_{i=1}^{2g}\lambda_i^{2k}=2, \qquad -2\leq 2k\leq n,\ k\neq 0, \tag{7.2k}$$

$$\iota_{2k-1}=\sum_{i=1}^{2g}\lambda_i^{2k-1}=0, \quad 0\leq 2k\leq n. \tag{7.2k-1}$$

By (7.i), (6.1) is $p_1=0$, and then (6.2) is $p_2=-1$. We use induction to show $p_l=0, 3\leq l\leq s$. Now (6.3) is $p_3=0$. Suppose $p_k=0$, $3<k<n$, then (6.$k+1$) has the form

$$\iota_{k+1}+(-1)\iota_{k-1}+(k+1)p_{k+1}=0.$$

If k is even, then we have $(k+1)p_{k+1}=0$, so $p_{k+1}=0$. If k is odd, then we have $2-2+(k+1)p_{k+1}=0$, so $p_{k+1}=0$.

It f is orientation preserving, then f^k is a self map of degree 1 for any integer k. Similarly we have

$$\iota_k=\sum_{i=1}^{2g}\lambda_i^k=2, \quad -2\leq k\leq n,\ k\neq 0. \tag{8.k}$$

By (8.k), (6.1) is $p_1=2$, and then (6.2) is $p_2=1$, and then $p_3=p_4=\cdots=p_s=0$ by (6.k), $k=3,\ldots,s$.

If $n\geq 2g$, since $2g\geq 3$, $p_{2g}=0$. This contradicts (3).

If $g>2$ and $n\geq 2g-2$, since $2g-2\geq 3$, we have $p_{2g-2}=p_{2g-1}=0$. By (5), we have $\iota_{-2}=0$, this contradicts (7.-2) and (8.-2).

Case 2: *Suppose f is a homeomorphism on N_q such that $f,\ldots,f^n$ are all fixed point free.* We show $n<q-1$ and $n<q-2$ if further $q>3$. Now $s=q-1$.

Since $H_2(N_g,Q)=0$, similarly we have

$$\iota_k=\sum_{i=1}^{2g}\lambda_i^k=1, \quad -1\leq k\leq n,\ k\neq 0. \tag{9.k}$$

By (9.k), (6.1) is $p_1=1$, and then $p_2=p_3=p_4=\cdots=p_s=0$ by (6.k), $k=2,\ldots,s$. If $n\geq q-1$, since $q>2$, $p_{q-1}=0$. This contradicts (3).

If $q>3$ and $n\geq q-2$, since $q-2\geq 2$, we have $p_{q-2}=0$. By (4), we have $\iota_{-1}=0$. This contradicts (9.-1).

We have finished the proof of Theorem 1. □

The author thanks the referee for his comments which improve the presentation of Step 2 significantly.

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